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On mass transports generated by tides and long waves

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For small-amplitude barotropic wave motion in a shallow fluid, Moore (1970) found that the associated mean mass transport is geostrophic, but otherwise arbitrary in the absence of friction. We show how weak friction, or starting the motion from rest, determines the mass transport by restricting circulation around closed geostrophic (f/h) contours. The resulting transport is quadratic in oscillatory quantities and depends on the friction type, but not on its (weak) magnitude. Comparison is made with earlier results in particular geometries. A tendency for anticyclonic circulation around shallow regions is found, and extends to large-amplitude oscillations where particle excursions exceed the topographic length scale. We suggest that numerical schemes for calculating tidal residuals should conserve mass and vorticity.

1. Introduction

Currents in the sea are usually dominated by fluctuating motions, and especially by regular tidal oscillations on many continental shelves. However, mean or residual currents (say the average over a tidal cycle) are important for the longer term movement of sea-floor sediments, fish larvae and indeed the sea-water itself.

Mean currents may arise from meteorological forcing, horizontal density gradients and externally imposed sea-surface slopes in limited-area models. These effects can usually be added (as indicated in §3 below) to that discussed here, namely, mean currents generated nonlinearly from oscillatory motion such as tides.

In broad terms, let $\mathbf{v}$ be an oscillatory current. Then the convective acceleration $\mathbf{v}.\nabla\mathbf{v}$, say, has a non-zero time average, and appears as a forcing term in the time-averaged momentum equation. This organized analogue of a turbulent Reynolds stress has been termed 'tidal stress' (e.g. Nihoul 1975) in a tidal context. It makes an important contribution to mean currents in (for example) the English Channel (Pingree & Maddock 1977), the southern North Sea (Prandle 1978) and over the Grand Banks (Loder 1980).

Laboratory experiments by Whitehead (1975) and Colin de Verdière (1979) have demonstrated mean currents generated by waves in rotating shallow water.

In a non-rotating context, wave-induced mean flows were first studied by Stokes (1847). They were calculated for small-amplitude oscillations by Longuet-Higgins (1953) in and close to one-dimensional thin boundary layers. Longuet-Higgins' (1970) calculation of circulation around islands utilizes this analysis for a side-wall boundary layer. Extensions to two horizontal dimensions were by Hunt & Johns (1963) for the bottom boundary layer and e.g. Liu (1977) for the free surface. Ianniello (1977, 1979) considered tidally-induced mean currents in essentially one horizontal dimension but with greater viscosity so that the bottom 'boundary layer' includes the whole depth.

On a deep rotating ocean, Ursell (1950; see also Hasselmann 1970) found that surface waves induce no mass transport. In rotating shallow water, mass transport in the bottom boundary layer has been calculated by Hunt & Johns (1963), Johns & Dyke (1972) and Lamoure & Mei (1977), *given* the mean current above, which remains to be determined. Neglecting viscosity entirely, Moore (1970) showed that the shallow water mass transport is horizontally non-divergent and follows geostrophic (f/h) contours. In a context uniform in one of the horizontal co-ordinates and including linear bottom friction, its *strength* was found by Huthnance (1973*a*) and Loder (1980). Oscillatory motion depending on both horizontal co-ordinates was considered by Ou & Bennett (1979), but their analysis (as opposed to numerical calculations) covered only the mean circulation along a straight coastline bounding water of uniform depth; linear bottom friction was again assumed. Stern & Shen (1976) considered slow oscillations of period t_0, where $t_0 \gg$ spin-up time $\gg$ inertial period.

For the small-amplitude oscillations common to all the above work, we find here that the strength of the mass transport can be evaluated from a simple constraint (3.8) of zero net frictional torque around closed fluid circuits of constant f/h. This holds for various friction types, and may also be applied to a context where the driving oscillatory flow evolves, as in the tidal spring-neap cycle or motion growing from rest.

Growing oscillations provide our link with other studies which regard the fluctuations as turbulent with no explicit representation of friction. Such an approach is probably more appropriate to the largely random (taking a broad view) quasi-geostrophic fluctuations in deep-sea and atmospheric motions. Turbulence and growing oscillations are distinguished from periodic waves by a time-increase of mean-square particle separations, with implications for eddy fluxes of the convected vorticity and thence for the mean-flow development (e.g. Rhines 1979). Thus, eddies may contribute to ocean circulation (e.g. Rhines & Holland 1979). Bretherton & Haidvogel (1976) found that the stream function for initially turbulent flow typically evolves to an almost steady smoothed version of the (slightly varying) depth, with anticyclonic circulation around shallows of large area and convection of potential vorticity without change across topography of small horizontal scale. In §§3 and 7 we find similar results for the mean flows induced by periodic motion.

After the equations of motion are presented in §2, the torque constraint on mean circulation is derived in §3. The inclusion of additional forcing by wind stress, horizontal density gradients and the tide generating potential is also discussed. The torque constraint is obtained again in §4 by a more physical argument using potential vorticity. A tendency for anticyclonic circulation around shallower areas is found. In §5 comparison is made with earlier results for one-dimensional features (Huthnance 1973*a*; Johns 1973) and a polar β-plane (Rhines 1976). Flows associated with oscillations of large amplitude are considered in §6; anticyclonic circulation around shallow regions is the rule in this case. In §7 the small amplitude theory is extended to regions of near-uniform f/h, to motion started from rest, and to lateral friction. Intermediate amplitudes are also considered, particularly over small-scale topography. The final discussion (§8) suggests that numerical schemes for modelling residual currents should conserve mass and vorticity and represent frictional forces realistically.

2. Formulation

We initially consider unforced 'hydrostatic' motion of a homogeneous, incompressible sea with surface elevation ζ above the still water level $z = 0$. The horizontal momentum equation is

$$\partial \mathbf{u}_H/\partial t + \epsilon \mathbf{u}.\nabla \mathbf{u}_H + f\mathbf{k}\wedge \mathbf{u}_H = -\nabla_H \zeta + F\partial \boldsymbol{\tau}/\partial z \quad (2.1)$$

where the suffix H denotes horizontal components and $\mathbf{k}$ is the vertical unit vector. Other quantities are defined and non-dimensionalized against various scales as follows:

horizontal co-ordinates, x, y: horizontal length scale L (the lesser of the topographic length scale or a wavelength);

upward vertical co-ordinate z: typical water depth h_0;

time t, inertial time scale f^{-1}: typical inertial time scale f_0^{-1};

velocity $\mathbf{u} \equiv (u, v, w)$: $(U, U, h_0 U/L)$ where U is a typical (oscillatory) current magnitude;

elevation ζ: $f_0 UL/$(gravitational acceleration g);

horizontal (stress/density) $\boldsymbol{\tau}$: typical (stress/density) τ_0.

The resulting non-dimensional parameters are (i) the Rossby number

$$\epsilon \equiv U/f_0 L$$

relating the magnitude U/f_0 of oscillatory particle excursions to the horizontal length scale L; (ii) the divergence parameter

$$D^2 \equiv f_0^2 L^2/gh_0$$

comparing the length scale L with the Rossby radius of deformation $(gh_0)^{\frac{1}{2}}/f_0$ and (iii) the parameter

$$F \equiv \tau_0/(h_0 U f_0)$$

measuring the relative importance of frictional stresses.

The continuity equation $\nabla.\mathbf{u} = 0$ integrates vertically to

$$D^2 \partial\zeta/\partial t + \nabla_H.[(h + \epsilon D^2 \zeta)\mathbf{v}] = 0 \quad (2.2)$$

exactly, when we apply the kinematic free surface condition and suppose no flow through the sea floor. The vertical average of the momentum equation (2.1) is (using continuity)

$$\partial \mathbf{v}/\partial t + \epsilon \mathbf{v}.\nabla_H \mathbf{v} + f\mathbf{k}\wedge \mathbf{v} = -\nabla_H \zeta - F\boldsymbol{\tau}_B/d. \quad (2.3)$$

Here d is the total water depth $h + \epsilon D^2 \zeta$, and $\boldsymbol{\tau}_B$ is the bottom friction stress. The horizontal velocity $\mathbf{u}_H = \mathbf{v} + \mathbf{u}_B$ is written as a combination of its vertical average

$$\mathbf{v} \equiv d^{-1} \int_{-h}^{\epsilon D^2 \zeta} \mathbf{u}_H \, dz$$

and a departure $\mathbf{u}_B$. A term

$$-\epsilon F d^{-1} \nabla_H . \left[\int_{-h}^{\epsilon D^2 \zeta} \mathbf{u}_B \mathbf{u}_B \, dz/F\right]$$

representing momentum flux in bottom (and surface) boundary layers has been omitted from the right hand side of (2.3); we reintroduce it with the wind stress and other forcing at the end of §3.

We consider in particular weak friction, i.e. small F; in a tidal context this in effect means a dissipation time of a day or more. F also measures the relative thickness of the bottom boundary layer, where $\mathbf{u}_B$ is significant; elsewhere in basically barotropic flow $\mathbf{u}_B$ is $O(F)$.

By cross-differentiating (2.3) a potential vorticity equation is obtained using (2.2):

$$(\partial/\partial t+\epsilon\mathbf{v}.\nabla_H)(\omega+\epsilon^{-1}f)/d = -F/d\,\mathbf{k}.\mathrm{curl}\,\boldsymbol{\tau}_B/d, \tag{2.4}$$

where $\omega \equiv \partial v/\partial x - \partial u/\partial y$ is the vertical vorticity component.

3. Small-amplitude oscillations

We write $\mathbf{v} = \mathbf{v}_1+\epsilon\mathbf{v}_2+\dots, \zeta = \zeta_1+\epsilon\zeta_2+\dots, \boldsymbol{\tau}_B = \boldsymbol{\tau}_{B1}+\epsilon\boldsymbol{\tau}_{B2}+\dots$, the Rossby number ϵ being small. The major contributions $\mathbf{v}_1$, ζ_1, etc. are supposed to be oscillatory with zero time average: $\overline{\mathbf{v}_1} = 0 = \overline{\zeta_1}$. Means $\mathbf{v}$, $\bar{\zeta}$ are included in $\epsilon\mathbf{v}_2$, $\epsilon\zeta_2$. This scaling accords with previous experience of rectified oscillatory motion, and is found below to be appropriate. Periodic motion is assumed so that $\overline{\partial X/\partial t} = 0$ for any flow quantity X. Overbars denote averaging over a period, and it is also convenient to write $\mathbf{v}_1^t$ for the indefinite time integral $\int^t \mathbf{v}_1 dt$, and $\mathbf{v}_{1t}$ for $\partial\mathbf{v}_1/\partial t$ etc.

To lowest order in ϵ, (2.3) gives

$$\partial\mathbf{v}_1/\partial t+f\mathbf{k}\wedge\mathbf{v}_1 = -\nabla_H\zeta_1-F\boldsymbol{\tau}_{B1}/h. \tag{3.1}$$

Solving (3.1) for $\mathbf{v}_1$ in terms of ζ_1 and $\mathbf{T}_1 \equiv (-\boldsymbol{\tau}_{B1t}+f\mathbf{k}\wedge\boldsymbol{\tau}_{B1})/h(f^2-\sigma^2)$, straightforward but lengthy algebra shows that

$$\begin{aligned}
&\overline{\mathbf{v}_1.\nabla_H\mathbf{v}_1}-f\mathbf{k}\wedge\overline{\mathbf{v}_1^t.\nabla_H\mathbf{v}_1}+\tfrac{1}{2}\nabla_H(\overline{\mathbf{v}_1.\nabla_H\zeta_1^t})\\
&= F\{\overline{\mathbf{T}_1.\nabla_H\mathbf{v}_1}+\overline{\mathbf{v}_1.\nabla_H\mathbf{T}_1}+f\mathbf{k}\wedge(\overline{\mathbf{T}_1.\nabla_H\mathbf{v}_1^t}+\overline{\mathbf{v}_1.\nabla_H\mathbf{T}_1^t})+\tfrac{1}{2}\nabla_H(\overline{\mathbf{T}_1.\nabla_H\zeta_1^t})\}+O(F^2)\\
&= F\{\tfrac{1}{2}\nabla_H(\overline{\mathbf{v}_1^t.\boldsymbol{\tau}_{B1}/h})+(\overline{\mathbf{k}.\mathrm{curl}\,\boldsymbol{\tau}_{B1}/h)(\mathbf{k}\wedge\mathbf{v}_1^t})\}+O(F^2).
\end{aligned} \tag{3.2}$$

Here σ is the oscillation frequency, and $\mathbf{T}_1$ merely a convenient intermediate notation. The lowest order F^0 balance in (3.2) was found by Moore (1970).

It is convenient to work with the Lagrangian-mean (mass-transport) velocity

$$\left.\begin{aligned}\epsilon\mathbf{U}_L &= \epsilon\overline{\mathbf{v}_2}+\epsilon\overline{\mathbf{v}_1^t.\nabla\mathbf{v}_1},\\ \text{'Lagrange} &= \text{Euler}+\text{Stokes'}\end{aligned}\right\} \tag{3.3}$$

(Longuet-Higgins 1953, 1969). By (2.2), following Moore (1970),

$$\nabla_H.[h\mathbf{U}_L] = O(\epsilon^2) \tag{3.4}$$

so that to sufficient accuracy $h\mathbf{U}_L$ has a stream function ψ: $\mathbf{U}_L = h^{-1}\nabla_H\psi\wedge\mathbf{k}$. Moore (1970) also deduced from (2.4) with zero right-hand side that

$$\mathbf{U}_L.\nabla(f/h) = O(\epsilon);$$

in fact, $O(\epsilon^2)$ holds here since there are no $O(\epsilon)$ contributions to $\mathbf{U}_L$. In our context,

$$\mathbf{U}_L.\nabla(f/h) = O(\epsilon^2, F). \tag{3.5}$$

Here and subsequently, the maximum of alternative estimates is implied. The constraints (3.4), (3.5) mean that $\mathbf{U}_L$ is geostrophic, analogous to Bretherton & Haidvogel's

(1976) conclusion for evolved turbulent flow around large-scale topography. Using (3.2, 3), the largest $O(\epsilon)$ terms of the *time-averaged* momentum equation (2.3) become

$$\nabla_H \bar{\zeta}_2 = \tfrac{1}{2}\nabla_H(\overline{\mathbf{v}_1 . \nabla\zeta_1^t}) - f/h\nabla_H\psi - F\overline{\boldsymbol{\tau}_B}/\epsilon d - F\{\tfrac{1}{2}\nabla_H(\overline{\mathbf{v}_1^t . \boldsymbol{\tau}_{B1}/h}) + (\mathbf{k} . \mathrm{curl}\,\overline{\boldsymbol{\tau}_{B1}/h) (\mathbf{k} \wedge \mathbf{v}_1^t})\}. \tag{3.6}$$

At this stage, ψ may be any function of $p \equiv f/h$. The lowest-order balance in (3.6) is

$$\nabla_H(\bar{\zeta}_2 - \tfrac{1}{2}\overline{\mathbf{v}_1 . \nabla_H \zeta_1^t} + \int^p p(d\psi/dp)dp) = 0, \tag{3.7}$$

which merely relates the mean surface elevation $\bar{\zeta}_2$ geostrophically to the mass transport stream function ψ. The mass transport ψ itself is *not* determined at this lowest order, but only weakly by the effects of friction, i.e. the $O(F)$ terms in (3.6). Although the $O(\mathrm{F})$ terms are small, they are the only ones having an integrated effect around any closed f/h contour. In other words, (3.6) really represents two balances of forces: a mean pressure balance (3.7), with terms $O(1)$ relative to mean pressures, and a weak $O(\mathrm{F})$ mean torque balance obtained by integrating (3.6) around any closed f/h contour:

$$\overline{\oint d^{-1}\boldsymbol{\tau}_B . \mathbf{dl}} = 0. \tag{3.8}$$

The circulation around any closed geostrophic contour adjusts so that the line integral of the depth-distributed stress $\boldsymbol{\tau}_B/d$ around the fluid circuit *as it moves* (a Langrangian viewpoint) averages to zero in time.

The strength of friction does not appear in (3.8). For practical application, (3.8) is written as

$$\oint h^{-1}\overline{(1 + \epsilon\nabla . \mathbf{v}^t)(\boldsymbol{\tau}_{B1} + \epsilon\boldsymbol{\tau}_{B2} + \epsilon\mathbf{v}^t . \nabla\boldsymbol{\tau}_{B1}) . (\mathbf{dl} + \epsilon\mathbf{dl} . \nabla\mathbf{v}^t)} = 0,$$

around the fixed geostrophic contour. This expansion takes sufficient account of the circuit motion for the lowest-order mean current estimate at $O(\epsilon)$. Hence any linear relationship between the current $\mathbf{v}$ and stress $\boldsymbol{\tau}_B$ results in a quadratic form for the mean current in terms of the oscillatory motion. Examples are given in §5, but we here note the simplest case $\boldsymbol{\tau}_B = k\mathbf{v}$ for which (3.8) becomes

$$\oint h^{-1}\overline{\mathbf{v}_2} . \mathbf{dl} = \oint h^{-1}(\overline{\mathbf{v}_1^t \wedge \mathrm{curl}\,\mathbf{v}_1 - \mathbf{v}_1\nabla . \mathbf{v}_1^t}) . \mathbf{dl} \tag{3.9}$$

or

$$\oint h^{-1}\mathbf{U}_L . \mathbf{dl} = \oint h^{-1}\{\overline{(\mathbf{dl} . \nabla\mathbf{v}_1) . \mathbf{v}_1^t} - \overline{(\nabla . \mathbf{v}_1^t)\mathbf{v}_1 . \mathbf{dl}}\} \tag{3.10}$$

completely determining $\mathbf{U}_L$ (and therefore $\overline{\mathbf{v}_2}$) by (3.4, 5). In our small amplitude (ϵ) weak friction (F) limit, the oscillations $\mathbf{v}_1$ are calculated merely from the linear inviscid shallow water equations (i.e. (3.1) without $F\boldsymbol{\tau}_{B1}$) and are assumed to be known.

This quadratic relationship, independent of the magnitude of weak friction, is analogous to that for the mean current adjacent to the bottom boundary layer in the non-rotating case (Longuet-Higgins 1953). However, rotation leads to uniformity with depth and so to a weak $O(F)$ determination throughout the depth rather than an $O(1)$ determination merely within $O(F)$ of the bottom. Moreover, the rotational constraint that $\mathbf{U}_L$ be geostrophic links together all points around an f/h contour in determining $\overline{\mathbf{v}_2}$ or $\mathbf{U}_L$, whereas the determination is local when rotation is absent.

The *form* of $\boldsymbol{\tau}_B$ may considerably affect the calculated circulation. This is illustrated

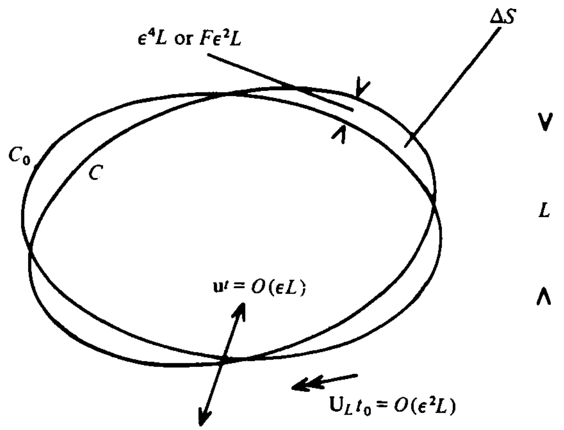

FIGURE 1. Small-ϵ closed circuit motion. Initially P = constant on C_0. Fluid on C_0 moves to C after one period t_0. Quantities are dimensional.

by the §5 examples and in §7 where $\boldsymbol{\tau}_B$ represents effects of horizontal viscosity and growing oscillations.

Under more general conditions the *dimensional* torque balance is

$$0 = -\tfrac{1}{2}\oint h\nabla_H\bar{\rho}\,.\,\mathbf{dl} - \oint h^{-1}\nabla_H\,.\left(\int_{-h}^{0}\overline{\mathbf{u}_B\,\mathbf{u}_B}\,dz\right).\,\mathbf{dl} + \oint h^{-1}\bar{\boldsymbol{\tau}}_w\,.\,\mathbf{dl} - \overline{\oint d^{-1}\boldsymbol{\tau}_B\,.\,\mathbf{dl}}. \tag{3.11}$$

The additional terms result from mean horizontal density gradients $\nabla_H\bar{\rho}$, mean momentum fluxes within the bottom (and/or surface) boundary layer, and the mean wind stress $\bar{\boldsymbol{\tau}}_w$. The essential linearity of the mean current problem (Heaps 1978) is thus illustrated for small ϵ (assuming mean currents small enough for insignificant interaction with each other or the oscillatory motion). Mass transports may be synthesized from 'tidal stress', density and wind stress contributions, as done by Prandle (1978), for example. A sea surface slope contribution requires either 'open' geostrophic contours crossing the region studied, or different conditions, e.g. stronger friction. The tide generating potential ζ_E and atmospheric pressure p_a contribute no torques, merely modifying the pressure balance (3.7) where $\zeta + p_a/\rho - \zeta_E$ replaces ζ.

4. An heuristic approach

We now obtain (3.8) by an order of magnitude argument, starting from the potential vorticity equation (2.4), viz.

$$d\,DP/Dt = -F\mathbf{k}\,.\,\mathrm{curl}\,\boldsymbol{\tau}_B/d \tag{4.1}$$

where $P \equiv (\omega + f/\epsilon)/d$ and D/Dt is the material derivative or rate of change following a fluid column.

Let C be any simple closed curve enclosing an area S and moving with the fluid. Then (4.1) gives

$$-F\,\overline{\oint_C d^{-1}\boldsymbol{\tau}_B\,.\,\mathbf{dl}} = \frac{D}{Dt}\int_S P\,d\,dA$$

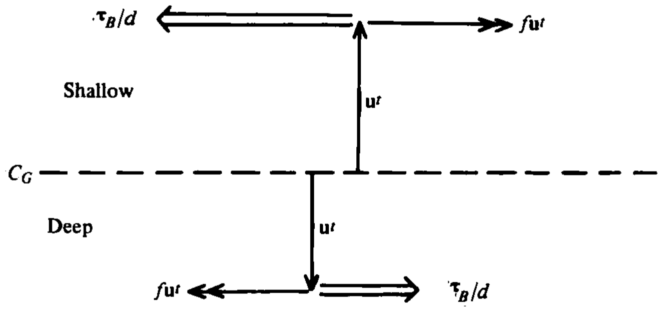

FIGURE 2. Net anticyclonic frictional force about shallow water: displacement (→), velocity (↠), force (⇒).

since the volume ddA of any fluid column remains constant: $D/Dt(ddA) = 0$. Averaging over one oscillation period t_0,

$$-\epsilon F \oint_C (d\epsilon)^{-1}\boldsymbol{\tau}_B.\mathbf{dl} = t_0^{-1}\int_{\Delta S} PddA \tag{4.2}$$

where ΔS is the area between the initial and final positions of C (figure 1). The integral over ΔS is well defined since ω is the same at both ends of the period t_0. We obtain (3.8) by showing that ΔS is small, and that the integrand $PddA$ integrates to almost zero by volume constraints.

Let C initially be a geostrophic contour C_G, apart from an $O(\epsilon^2, F)$ displacement. C moves on average with the mass transport velocity $\epsilon\mathbf{U}_L$, and therefore parallel to itself with error $O(\epsilon^2, F)$. After one period the dimensional normal displacement is $O(\epsilon U(\epsilon^2, F)/f_0)$, i.e. non-dimensionally

$$\Delta S = O(\epsilon^2, F)\epsilon^2.$$

Around C_G, P is nearly uniform. Individual fluid columns follow C_G with error $O(\epsilon^2, F)$ around a circuit, owing to the direction of $\mathbf{U}_L$, and during a circuit the value of P changes by the average rate $F\epsilon$, from (4.1), multiplied by the time ϵ^{-2}. Hence P equals a uniform value P_G on C_G with error only $O(\epsilon, F/\epsilon)$, and we can find a nearby curve C_0 on which $P = P_G$ exactly. C_0 and C_G are separated by at most $O(\epsilon^2, F)$. Let $C = C_0$ initially. Then $Pd = P_G d$ on C_0, $[P_G + O(F\epsilon)]d$ on the same fluid circuit after one cycle and hence in ΔS; (4.2) gives

$$\overline{\oint_C (d\epsilon)^{-1}\boldsymbol{\tau}_B.\mathbf{dl}} = -P_G(t_0\epsilon F)^{-1}\int_{\Delta S} ddA + O(\epsilon^2, F)\epsilon^2 F\epsilon/F\epsilon.$$

Volume conservation requires $\int_{\Delta S} ddA = 0$. Since C_0 and C_G are close,

$$\overline{\oint_C (d\epsilon)^{-1}\boldsymbol{\tau}_B.\mathbf{dl}} = O(\epsilon^2, F) \tag{3.8}$$

when C is initially a geostrophic contour C_G.

To summarize, the mass transport is so constrained by f/h contours and volume conservation that essentially the same fluid circuit keeps its position around a closed

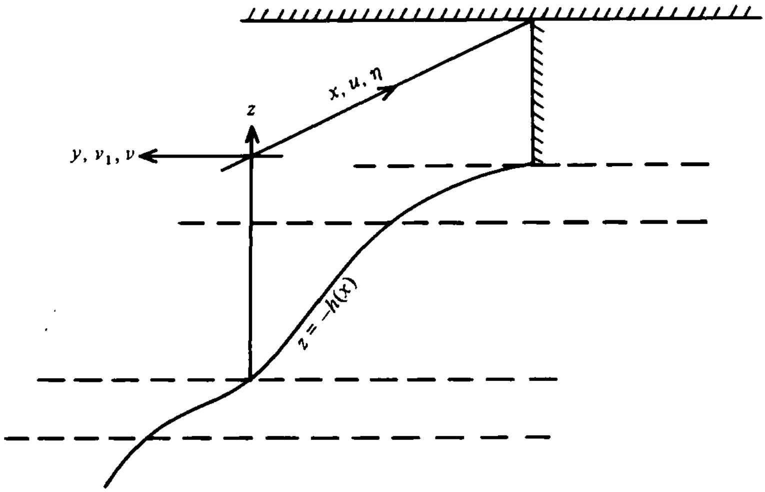

FIGURE 3. One-dimensional context. Bottom $z = -h(x)$; depth contours (– – –); coast (////) (divergent flow only).

f/h contour. Hence the mean circulation has time to settle down to a value for which there is no mean torque on the circuit.

Along the sloping sides of an area of shallower water, (3.8) tends to imply an anticyclonic circulation around that area. For uniform f, consider a fluid column in C, where initially C is a closed depth contour C_G around a sandbank or other shallow area (figure 2). At the time of the column's maximum displacement up the slope into shallower water, the Coriolis force has made its total contribution to the oscillatory cyclonic velocity component. Consequently there is an anticyclonic frictional force, which is more strongly felt by the column for being in shallower water. Half a cycle later, the corresponding cyclonic frictional force is less strongly felt for being in deeper water. This anticyclonic resultant of oscillatory frictional forces will generate an anticyclonic mean circulation until balanced by the latter's associated frictional drag.

5. Examples

(a) One-dimensional features with various forms of friction

The analysis for non-divergent motion in Huthnance (1973*a*) with $\boldsymbol{\tau}_B = k\mathbf{v}$ gives

$$\overline{h^{-1}v}^{c} = 0 \tag{5.1}$$

where $h = h(x)$ (figure 3), f is uniform and the average is taken following a fluid column. This is clearly the form taken by (3.8) omitting integration over the uniform conditions along the geostrophic (i.e. depth) contour. Hence for an 'incident' current $Re(u_\infty, v_\infty)\exp(i\sigma t)$ where $h = h_\infty$ (uniform), the Eulerian-mean current is

$$\bar{v} = -h^{-1}dh/dx(f\overline{\eta^2} - \overline{\eta v_1}) \tag{5.2}$$

where $\eta \equiv u_1^t$ is the local x displacement, amplitude $|u_\infty|h_\infty/h\sigma$. In this and the following one-dimensional examples, the Lagrangian mean V_L is $\bar{v} - \overline{f\eta\partial\eta/\partial x}$. Hence $\bar{v}$ and

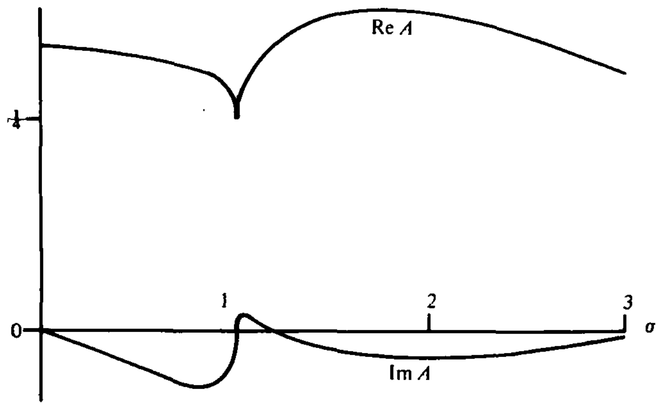

FIGURE 4. The mean current coefficient $A(\sigma)$ in §5(a), for uniform viscosity. $A = \frac{1}{2}$ for $\boldsymbol{\tau}_B = \mathbf{v}$.

V_L are quadratic in the oscillatory current, independent of the weak friction, and usually (especially $\bar{v}$) anticyclonic around shallower water since if $h \ll h_\infty$ and $v_\infty \lesssim u_\infty$, then $v_1 \doteq -f\eta$ in (5.2). These properties are shared by the widely used form $\boldsymbol{\tau}_B = C|\mathbf{v}|\mathbf{v}$ for which

$$\overline{h^{-1}|\mathbf{v}|\mathbf{v}}^{c} = 0, \tag{5.3}$$

$$\bar{v} = -h^{-1}dh/dx\overline{[f\eta^2(2v_1^2+u^2)-\eta v_1(v_1^2+2u^2)]/|\mathbf{v}|}/\overline{(2v_1^2+u^2)/|\mathbf{v}|} \tag{5.4}$$

replace (5.1) and (5.2) respectively. However, if (for example) the oscillatory v_1 is locally zero, then

$$\bar{v} = -f\overline{\eta^2}h^{-1}dh/dx \quad \text{(linear friction)},$$

$$\bar{v} = -\tfrac{2}{3}f\overline{\eta^2}h^{-1}dh/dx \quad \text{(quadratic friction)};$$

the two friction laws imply quantitatively different mean flows.

For divergent flow in uniform depth against a coast (figure 3) with $\boldsymbol{\tau}_B = k\mathbf{v}$, (5.1) becomes $\overline{d^{-1}v}^{c} = 0$. Hence

$$\bar{v} = f\frac{\partial}{\partial x}\overline{\eta^2} \tag{5.5}$$

replaces (5.2) if we assume zero pressure gradient along the coast so that $v_1 = -f\eta$.

If friction is modelled by an eddy viscosity $\nu_0 K(\xi)$, then we take $F \equiv (\nu_0/f_0 h_0^2)^{\frac{1}{2}}$; $\xi \equiv (z+h)/F$ is a scaled vertical co-ordinate for the bottom boundary layer where $\mathbf{u}_B$ is significant. The torque balance (3.8) must be modified to include mean momentum fluxes in the bottom boundary layer as in (3.11). The bottom stress $\boldsymbol{\tau}_B$ is generally a rather complicated functional of $\mathbf{v}$ as discussed by Johns & Dyke (1972), and we omit details here, although there is a simple linear contribution from the mean flow $\bar{\mathbf{v}}$. Replacing (5.5),

$$\bar{v} = f[A(\sigma)\eta^*\eta_x + A^*(\sigma)\eta\eta_x^*]$$

where η now represents a complex amplitude. $A = \frac{1}{2}$ would recover (5.5), but (for example) uniform viscosity $K \equiv 1$ actually results in $A(\sigma)$ as shown in figure 4. At low frequencies ($F \ll \sigma \ll f$), $A = 27/80$ so that

$$\bar{v} = \tfrac{27}{40}f\frac{\partial}{\partial x}\overline{\eta^2}.$$

Stern & Shen (1976) consider the case (not treated here) of oscillations so slow that spin-up is always complete, i.e. $\sigma \ll F \ll f$.

The eddy viscosity forms $K(\xi) = \xi^p$ pass from uniform viscosity at $p = 0$ to the linear increase with distance from the bottom characteristic of a logarithmic boundary layer at $p = 1$. The latter form has been extended to model the rotary spiral as well (e.g. Kagan 1972). Then for small F the additional boundary layer mean momentum flux term from (3.11) may be neglected, and (5.5) applies as the bottom stress approximates $\boldsymbol{\tau}_B = k\mathbf{v}$. More details are included in Huthnance (1973*b*). If the 'constant' ν_0 is proportional to the instantaneous velocity $|\mathbf{v}|$ outside the boundary layer, then the eddy viscosity $\nu_0\xi$ should yield $\boldsymbol{\tau}_B = c|\mathbf{v}|\mathbf{v}$ approximately.

A one-dimensional context cannot strictly exist, but forms a good approximation to circular symmetry if the radius greatly exceeds the 'x' length scale. It is also applicable to linear features such as sandbanks (Huthnance 1973*a*) where most of the integration in the circulation constraint (3.8) is along two straight-line segments. If the context is only *locally* one-dimensional, then the geostrophically guided mean current is subject to external influence (Johns 1973) since the frictional control is weak. For example, an endwall at $y =$ const (figure 3) would prevent mass transport along the depth contours.

(*b*) *A polar β-plane*

Rhines (1976) considered mean flows associated with non-divergent oscillations in water of uniform depth when f decreases linearly from the centre of rotation. In his §8*D*, a simple bottom friction $\boldsymbol{\tau}_B = D\mathbf{v}$ is used (our notation) so that (3.9) can be applied. For the oscillatory motion we write $-\eta \equiv$ radial displacement; since $\mathbf{k}\,.\,\mathrm{curl}\,\mathbf{v}_1 = -\beta\eta$ (Rhines' equation (35)) we have a zonal mean flow

$$\bar{u} = -\beta\overline{\eta^2}.$$

This agrees with Rhines' equation (38), case (*c*) which matches our case of periodic oscillations. In §7(*b*) below we also match Rhines' (1976) result

$$\bar{u} = -\tfrac{1}{2}\beta\overline{\eta^2},$$

for inviscid motion starting from rest. The negative signs imply anticyclonic flow. Both these formulae are relevant to Whitehead's (1975) experiment, as discussed by Rhines (1976).

(*c*) *Small depth variations in a rectilinear current*

Consider a rectilinear oscillatory current $\mathbf{v}_1 \simeq U(t)\mathbf{l}$ (associated with a Kelvin wave) slightly perturbed by small depth variations of length scale much less than a tidal wave length. For linear bottom friction and uniform f, (3.9) gives

$$\oint\bar{\mathbf{v}}\,.\,\mathbf{dl} = -h^{-1}\oint\overline{(U^t)^2}f(\mathbf{l}\,.\,\mathbf{n})^2\partial h/\partial n\,ds \tag{5.6}$$

where we have written $\nabla\,.\,\mathbf{v}_1^t = -h^{-1}\mathbf{v}_1^t\,.\,\nabla h$, and $\omega_1 = f/h\mathbf{v}_1^t\,.\,\nabla h$ by (2.4). 'On average' $\partial h/\partial n$ is uncorrelated with $(\mathbf{l}\,.\,\mathbf{n})^2$ which averages $\frac{1}{2}$, so that the mean circulatory current is

$$\bar{v} = \overline{(U^t)^2}f|\nabla h|/2h$$

and is anticyclonic around shallow water regions. Hence the mean vorticity is proportional to (oscillatory displacement)^2, $|\nabla h|/h$, and (topographic length scale)$^{-1}$ 'on

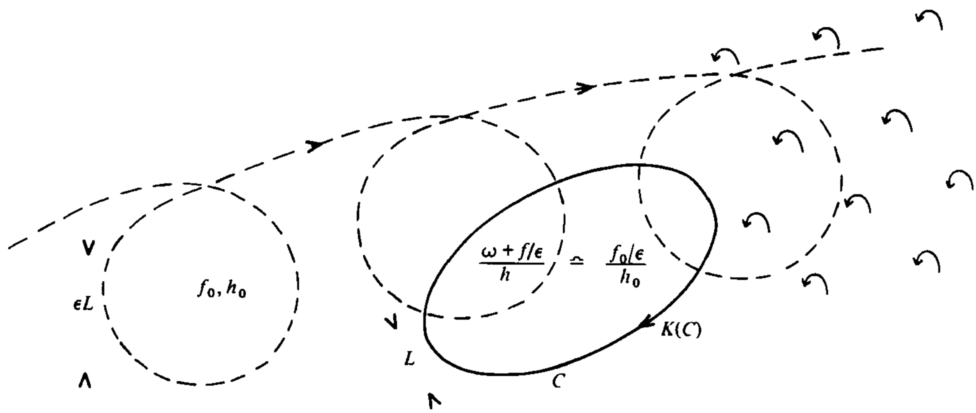

FIGURE 5. Isolated feature beneath large amplitude oscillations. Particle path (– – –); weak 'downstream' vorticity (↶) totals $-K(C)$.

average'. This agrees with Zimmerman's (1978) result for small-amplitude oscillations in the context of small random topography.

6. Large-amplitude oscillations

Consider (figure 5) an isolated topographic feature, such as a sandbank, bounded by a closed curve C of horizontal extent less than an oscillatory particle excursion, i.e. $\epsilon \gtrsim 1$. We take the Coriolis parameter f to be essentially uniform, as also is the depth in the region R outside C. Hence the effective length scale in R is much larger and we take $\epsilon_R \ll 1$.

The potential vorticity equation (2.4) with $\boldsymbol{\tau}_B = k\mathbf{v}$ is

$$(\partial/\partial t + \epsilon \mathbf{v}.\nabla_H)(\omega + f/\epsilon)/h = -F/h\,\mathbf{k}.\text{curl}\ \mathbf{v}/h \tag{6.1}$$

(In fact most of this section is clearly independent of the particular friction type.) We have assumed that surface elevations are a small fraction of the total depth ($\epsilon D^2 \ll 1$), writing h for d.

In R, where f and h are essentially constant, (6.1) gives

$$\frac{D}{Dt}\frac{\omega}{h} = -\frac{F}{h}\frac{\omega}{h}$$

so that for each fluid column ω spins down to zero. For fluid columns approaching C from R we then have

$$(\omega + f/\epsilon)/h = (f/h\epsilon)_C. \tag{6.2}$$

More detailed discussion of the conditions for (6.2), if f and h are not uniform in R, is given in Huthnance (1973*b*), but the best justification is *a posteriori* from the resulting flow field. The occurrence of spin down is independent of the particular form of bottom friction.

Since $\epsilon \gtrsim 1$, we can expect that fluid columns stay inside C only for a few oscillations, or for only a small fraction of the time. Then by (6.1) the changes in potential vorticity $(\omega + f/\epsilon)/h$ are $O(F)$ and can be neglected to lowest order. Hence (6.2) also holds within C. We clearly exclude the small ϵ phenomenon of a Taylor column or the

Lagrangian-mean flow around f/h contours discussed in §§3–5. Again the best justification is *a posteriori*. As with small ϵ, we note the analogy with Bretherton & Haidvogel's (1976) conclusion for evolved turbulent flow, here over small-scale topography.

The mean current is determined by (6.2) and the time-averaged continuity equation (2.2), viz.

$$\nabla_H \cdot (h\bar{v}) = 0 \tag{6.3}$$

provided $\epsilon D^2 \ll 1$ as before. These are simply the equations governing a steady current which conserves potential vorticity, and are treated by Batchelor (1967, §7.7). In particular, the circulation around any closed curve Γ enclosing a domain D near or inside C is

$$K(\Gamma) = \iint_D f/\epsilon(h/h_C - 1)dA; \tag{6.4}$$

$K(\Gamma)$ is anticyclonic around shallow areas. Further, the mean vorticity is proportional to $|\nabla h|/h$ and the topographic length scale, contrasting with the small-amplitude result in §5(d). The latter's proportionality to (oscillatory displacement)^2 is replaced by (topographic length scale)^2 when ϵ reaches $O(1)$.

The circulation (6.4) cannot persist far from the small curve C since any friction would induce large torques. In fact, (6.1) times h integrated over the area within C implies a vorticity generation rate $-FK(C)/h_C$ there. Since we consider a 'steady state' with no vorticity build-up within C, this quantity is advected outside to R. It decays in R with time-constant F/h_C by (6.1), so that the total vorticity in R is $-K(C)$. This just cancels the total vorticity $K(C)$ within C. There is no circulation around any circuit in R enclosing all the advected vorticity $-K(C)$ as well as C. Since F is small, the total vorticity $-K(C)$ is smeared out thinly over a large area of R, and to lowest order may be ignored. This paragraph is really a justification for using (6.2) near C despite the attendant spurious circulations around distant curves in R enclosing C.

Similarly (Huthnance 1973*b*), for a steep-sided *island* (6.1) leads to

$$\oint h^{-1}\bar{\mathbf{v}}.\mathbf{dl} = 0, \quad \oint \mathbf{v}.\mathbf{dl} = \text{constant}$$

where the closed path of integration around the island is taken immediately outside the steep slope. If instead the island has a shore with absolute slope $O(h_0/L)$, i.e. $O(1)$ after scaling, then

$$\oint h^{-1}\mathbf{v}.\mathbf{dl} \quad \text{is finite}$$

when the closed path of integration tends to the island shoreline.

On account of (6.2), the equations of motion (2.2, 3) take the form

$$\nabla.(h\mathbf{v}) = 0, \quad \partial\mathbf{v}/\partial t + f_M\mathbf{k}\wedge\mathbf{v} = -\nabla Z$$

if we neglect $O(F, \epsilon D^2, D^2)$ and all forcing. Here $f_M = h(f/h)_C$, $Z = \zeta + \frac{1}{2}\epsilon\mathbf{v}^2$. These equations are linear in the flow variables $\mathbf{v}$, Z despite the inherent nonlinearity when $\epsilon \gtrsim 1$; the surface elevation is related non-linearly to the currents through Z. They also describe linearized non-divergent flow over a variable depth h when the Coriolis parameter is f_M. For oscillations $\operatorname{Re} u \exp(i\sigma t)$ we may write

$$\mathbf{u} = h^{-1}\nabla\psi \wedge \mathbf{k} \quad \text{where} \quad \nabla.(h^{-1}\nabla\psi) = 0$$

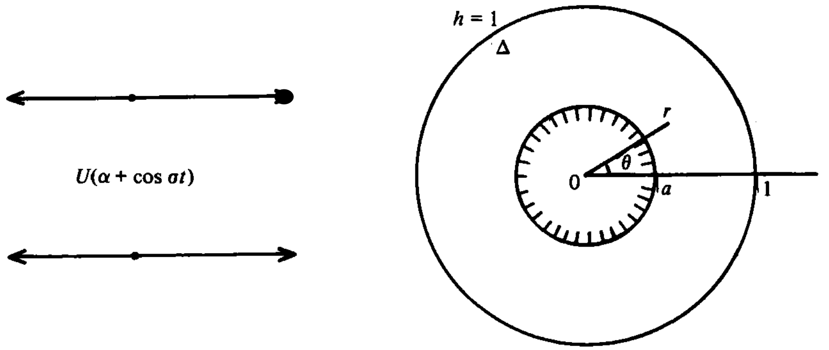

FIGURE 6. Island-shelf geometry in large incident current.

since f_M/h is uniform. The oscillatory motion has zero vorticity, as implied by (6.2), and on our small (non-divergent) length scale can merely interpolate inside and near C between the externally imposed oscillations. Gravity waves are filtered out by the non-divergence, and topographic Rossby waves are impossible because the 'base-state' potential vorticity $f_M/h = (f/h)_C$ is uniform.

As an example (figure 6) we consider the motion near a circular island $r < a$ (using plane polar co-ordinates r, θ). There is a shelf of depth $h = \Delta (a < r < 1)$. Elsewhere $h = 1$; $f = 1$ everywhere. We suppose a uniform rectilinear oscillatory current and steady flow $U(\alpha + \cos\sigma t)(\cos\theta, -\sin\theta)$ far from C: $r = 1$. Matching ψ and $h^{-1}\partial\psi/\partial r$ across $r = 1$ we have

$$\psi = \begin{cases} U\sin\theta(\alpha + \cos\sigma t)(Dr - E/r) - \Delta A\ln r & (r < 1), \\ U\sin\theta(\alpha + \cos\sigma t)(r - B/r) - A\ln r & (r > 1), \end{cases}$$

where $\{B, D, E\} = (1 + \Delta + (1 - \Delta)a^2)^{-1}\{1 - \Delta + (1 + \Delta)a^2, 2\Delta, 2\Delta a^2\}$, in addition to the mean azimuthal flow

$$\bar{v} = -f(1 - \Delta)/2\epsilon \begin{cases} r & (a < r < 1), \\ 1/r & (r > 1). \end{cases}$$

The circulation constraint at the island determines $A = f(1 - \Delta)a^2/2\epsilon$. The total anticyclonic circulation $\oint \mathbf{u}.\mathbf{dl}$ around the circle at radius r is therefore

$$f(1 - \Delta)\pi/\epsilon \begin{cases} r^2 - a^2 & (r < 1), \\ 1 - a^2 & (r > 1), \end{cases}$$

although, as discussed above, this should decrease slowly to zero for sufficiently large r.

7. Other extensions

(a) *Small gradients of f/h*

If gradients of f/h are small, $\mathbf{U}_L$ is less constrained to follow geostrophic contours. Nevertheless, the time-averaged momentum balance (3.6) still holds, and its lowest order (pressure) balance is a simpler form of (3.7):

$$\nabla_H[\bar{\zeta}_2 - \tfrac{1}{2}\overline{\mathbf{v}_1 . \nabla_H \zeta_1^t} + \psi f/h] = 0.$$

Hence the mean surface elevation is related to the mass transport stream function ψ. However, ψ is now arbitrary. As before, (3.6) also contains a weak torque balance, which is now much more widely applicable. Indeed, when f/h is uniform, (3.11) (and (3.8) when appropriate) applies around all closed fluid circuits. Equivalently, we take the curl of (3.6) for a mean vorticity equation; assuming $\bar{\boldsymbol{\tau}}_{B2}$ depends linearly on $\bar{\mathbf{v}}$ and $\mathbf{U}_L$,

$$\nabla_H . (h^{-2}\nabla_H \psi) = \text{quadratic form in known oscillatory quantities.}$$

This clearly determines ψ subject to the natural condition of specifying ψ on the boundary. However, (3.8) is a more direct statement of the physics and in simple geometries (cf. §5) can prove an easier basis for calculation.

An intermediate case occurs if $\nabla_H(f/h) = O(F)$ so that the guidance of $\mathbf{U}_{\mathrm{L}}$ by geostrophic contours is weak and comparable with the frictional torques. For linear bottom friction, curl (3.6) gives

$$\nabla_H . (h^{-2}\nabla_H \psi) + \mathbf{k} . (\nabla_H \psi \wedge F^{-1}\nabla_H f/h) = \text{quadratic form in known oscillatory quantities.}$$

It is easily shown that specifying ψ on the boundary ensures uniqueness of any solution. The equivalent of (3.8) is

$$-\overline{\oint d^{-1}\boldsymbol{\tau}_B . \mathbf{dl}} = \oint F^{-1}(f/h - \langle f/h \rangle)\nabla_H \psi . \mathbf{dl},$$

where $\langle f/h \rangle$ is the average of f/h around the circuit. Hence the net frictional torque around the fluid circuit provides for the export of potential vorticity in the Lagrangian-mean current. However, this circuit integral is no longer a practical basis for calculation.

(b) *Inviscid motion of varying amplitude*

Hitherto, the term $\boldsymbol{\tau}_B/d$ has been interpreted as a bottom stress. However, as it stands in the momentum equation (2.3), it may represent any of several weak constraints on the motion and circulation. One of particular interest is oscillatory motion growing slowly from rest. We represent this by

$$\{\mathbf{v}(\mathbf{x}, t), \zeta(\mathbf{x}, t)\} \exp(\mu t),$$

where μ is small. Then $F = \mu/f_0$ and $\boldsymbol{\tau}_B/d$ represents $\hat{\mathbf{v}} \equiv \mathbf{v}_1 + 2\epsilon\mathbf{v}_2$, if $\epsilon \ll 1$ and $D^2 \ll 1$ so that the non-divergent approximation holds. The circulation result (3.8) becomes a modified form of Kelvin's circulation theorem,

$$\overline{\oint \hat{\mathbf{v}} . \mathbf{dl}} = 0. \tag{7.1}$$

Note that $\hat{\mathbf{v}} \neq \mathbf{v}$. From (7.1),

$$\oint_{C_G} \bar{\mathbf{v}}_2 . \mathbf{dl} = \frac{1}{2}\oint_{C_G} h\overline{(\mathbf{v}_1^\dagger . \mathbf{n})^2}\partial/\partial n(f/h)dl, \tag{7.2}$$

which implies anticyclonic mean circulations around shallow regions of fluid. This occurs because fluid columns entering C_G gain anticyclonic relative vorticity (by conservation of potential vorticity), whilst those displaced outwards across C gain cyclonic relative vorticity.

Writing $\mathbf{v}_1^\dagger . \mathbf{n} = \eta$ and $\partial f/\partial n = -\beta$, (7.2) immediately gives Rhines' (1976) result $\bar{\mathbf{u}} = -\frac{1}{2}\beta\overline{\eta^2}$ for the polar β-plane (5(*b*): h is uniform, C_G is a latitude circle and zonal averages are taken). Colin de Verdière (1979) observed such a mean flow experimentally. His calculations further suggest that this zonal mean current has an initial growth rate $\mu^{\frac{1}{2}}\epsilon U \gg 2\mu\epsilon U$ if the (modulated) waves already exist without it.

If D^2 is not small, the Lagrangian-mean velocity is slightly divergent: by (2.2)

$$\nabla_H \cdot (h\mathbf{U}_L) = FD^2\{\nabla_H \cdot (\overline{\zeta^t \mathbf{v}}) - 2\bar{\zeta}_2\}$$

replacing (3.4). The additional terms on the right-hand side of (3.8) or (3.11) are then

$$-F^{-1}\oint f\mathbf{k}\wedge\mathbf{U}_L \cdot d\mathbf{l} = -F^{-1}f/h\oint h\mathbf{U}_L \cdot \mathbf{n}dl = D^2 f/h\{\oint \overline{\zeta \mathbf{v}^t} \cdot \mathbf{n}dl + 2\int\bar{\zeta}_2 dA\};$$

$$\oint \overline{\mathbf{v}}_2 \cdot d\mathbf{l} = \tfrac{1}{2}\oint h(\overline{\mathbf{v}^t \cdot \mathbf{n}})^2 \partial/\partial n(f/h)dl + (f/h)D^2\int\bar{\zeta}_2 dA \qquad (7.3)$$

replaces (7.2). The integral of $\bar{\zeta}_2$ is taken over the area S_G enclosed by the closed f/h contour C_G, and represents a mean column stretching within C_G. This has a direct effect on the circulation around C_G by conservation of potential vorticity. By (3.7), $\int\bar{\zeta}_2$ is a double integral of the mass transport velocity $d\psi/dp$, so that (7.3) is a second order ordinary differential equation in $p \equiv f/h$ for $\int\int p d\psi/dp$.

As an example we consider a β-plane ($f = \beta y$) of uniform depth $h = 1$, and suppose all quantities independent of x. Then the line integral in (7.3) is taken parallel to the x axis, but may be dispensed with on account of the uniformity in x (cf. 5(a)). The area integral implies integration in y. Writing $\eta = v^t$ and using (3.7), (7.3) becomes

$$\chi_{yy} - (\beta D y)^2\chi = N$$

where

$$N = \tfrac{1}{2}\beta y^2 D^2 \int^y \overline{\eta\zeta_{y'}}\, dy' + y(\overline{\eta u_y} - \tfrac{1}{2}\beta\overline{\eta^2}).$$

The value of N is known from the oscillatory motion, and the solution $\chi \equiv \int^y\int^{y'} y''(d\psi/dy'')dy''dy'$ determines both $\bar{\zeta}_2 = -\tfrac{1}{2}\overline{\eta\zeta_y} - \beta d\chi/dy$ and $U_L = y^{-1}d^2\chi/dy^2$.

Since the results of this section are independent of the small growth rate and even of its sign, we can consider any slow change of amplitude. Then the change of mean current is given by the change in the right-hand side of (7.2), say. Hence the spring-neap tidal cycle (for example) entails an associated fortnightly (MSf) variation in 'mean' currents and sea surface level. In practice, shallow-sea friction is likely to be strong enough to play a role; a decay time of a few days or less ensures this. Then the 'mean' currents are determined by a linear combination of (3.8) and (7.2), in proportion to the friction strength and current rate of change respectively. A fortnightly variation in 'mean' currents and surface elevation still ensues.

(c) *Lateral stress*

If the right-hand side of the momentum equation (2.3) includes weak viscous terms

$$(\nu_0/f_0L^2)\nabla_H \cdot (\nu\nabla_H \mathbf{v})$$

then we can define $F \equiv \nu_0/f_0L^2$ (the horizontal Ekman number) and use $\boldsymbol{\tau}_B/d$ to represent $-\nabla_H \cdot (\nu\nabla_H\mathbf{v})$. Hence (3.8) becomes

$$\oint\overline{\nabla_H \cdot (\nu\nabla_H\mathbf{v}) \cdot d\mathbf{l}} = 0. \qquad (7.4)$$

In general this cannot be made any more explicit for $\overline{\mathbf{v}}$, and merely illustrates the same constraint on the circulation that the torque around a material fluid circuit must average zero. However, in a one-dimensional context with $\partial/\partial y = 0$ (cf. 5(*a*), figure 3), we omit the line integral. Then for uniform ν,

$$\bar{v}_{xx} = -\overline{\eta v_{xxx}}:$$

$$\bar{v} = Ax + B - \left[h^{-1}dh/dx + \tfrac{3}{2}h^2 \int^x (h^{-2}dh/dx)^2\right] f\eta^2$$

replaces (5.2) in the non-divergent context of Huthnance (1973*a*). The constants A and B must be determined by lateral boundary conditions, but the term in dh/dx shows a tendency for anticyclonic circulation around shallow water-regions.

(*d*) *Intermediate amplitudes*

Andrews & McIntyre (1978) have introduced a generalized definition of Lagrangian-mean velocity, $\bar{\mathbf{u}}^L$, valid for oscillations of finite amplitude. Their corollary III regarding circulation becomes

$$\frac{d}{dt}\oint_\Gamma (\bar{\mathbf{u}}^L - \mathbf{p} + \epsilon^{-2} f\mathbf{k}\wedge\mathbf{x}).\mathbf{d}s = -F\oint \mathbf{d}s.\{\overline{\boldsymbol{\tau}_B/d}^L + \overline{(\nabla\boldsymbol{\xi}).(\boldsymbol{\tau}_B/d)^l}\} \tag{7.5}$$

where a closed curve Γ moves with the velocity $\bar{\mathbf{u}}^L$ and the 'pseudomomentum' $\mathbf{p}$ is a known quadratic function of the oscillatory displacements $\boldsymbol{\xi}$. Although our results do not follow simply from (7.5), this equation does nicely demonstrate the transition, as ϵ increases, from friction-dominated to vorticity-conserving mean flows.

For small ϵ the right-hand side of (7.5) is $-F\oint \overline{d^{-1}\boldsymbol{\tau}_B}.\mathbf{ds}$. Hence (3.8) implies that the Lagrangian-mean flow is so closely bound to f/h contours that the left-hand side of (7.5) can be neglected.

For $\epsilon \gtrsim 1$, small F implies near-conservation of potential vorticity, effectively neglecting the right-hand side of (7.5). The Lagrangian-mean flow is then (almost by assumption) not at all bound by f/h contours.

For intermediate ϵ, both sides of (7.5) play a role. The mean circulation around fluid circuits moving with the Lagrangian-mean velocity evolves under the influence of frictional torques.

(*e*) *Small depth variations*

We can find the Eulerian-mean current for intermediate ϵ when the depth variations are small. This is the case considered by Zimmerman (1978, 1979), and can be made more explicit when friction is weak.

Since the depth variations are small, $O(\delta)$ where $\delta \ll 1$, we can write

$$h = h_0 + \delta h_1(\mathbf{x}) + \dots,\quad \mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}_1(\mathbf{x}) + \dots,\quad \zeta = \zeta_0 + \epsilon_0\delta\zeta_1(\mathbf{x}) + \dots,$$
$$\boldsymbol{\tau}_B = \boldsymbol{\tau}_0 + \delta\boldsymbol{\tau}_1(\mathbf{x}) + \dots$$

where $\mathbf{v}_0$, ζ_0, $\boldsymbol{\tau}_0$ satisfy (2.2, 3) exactly for $h = h_0$ but, like h_0, vary only slowly, by $O(\epsilon_0)$, on the present short length scale L of the small depth changes. We can also assume $D^2 = O(\epsilon_0^2)$. Then at $O(\delta)$ the continuity equation yields

$$\nabla.\mathbf{v}_1 = -h_0^{-1}\mathbf{v}_0.\nabla h_1 \tag{7.6}$$

and the vorticity equation (2.4) yields

$$D/Dt(\omega - fh_1/h_0) = -F\mathbf{k}.(h_0^{-1}\operatorname{curl}\boldsymbol{\tau}_1 - h_0^{-2}\nabla h_1\wedge\boldsymbol{\tau}_0) \tag{7.7}$$

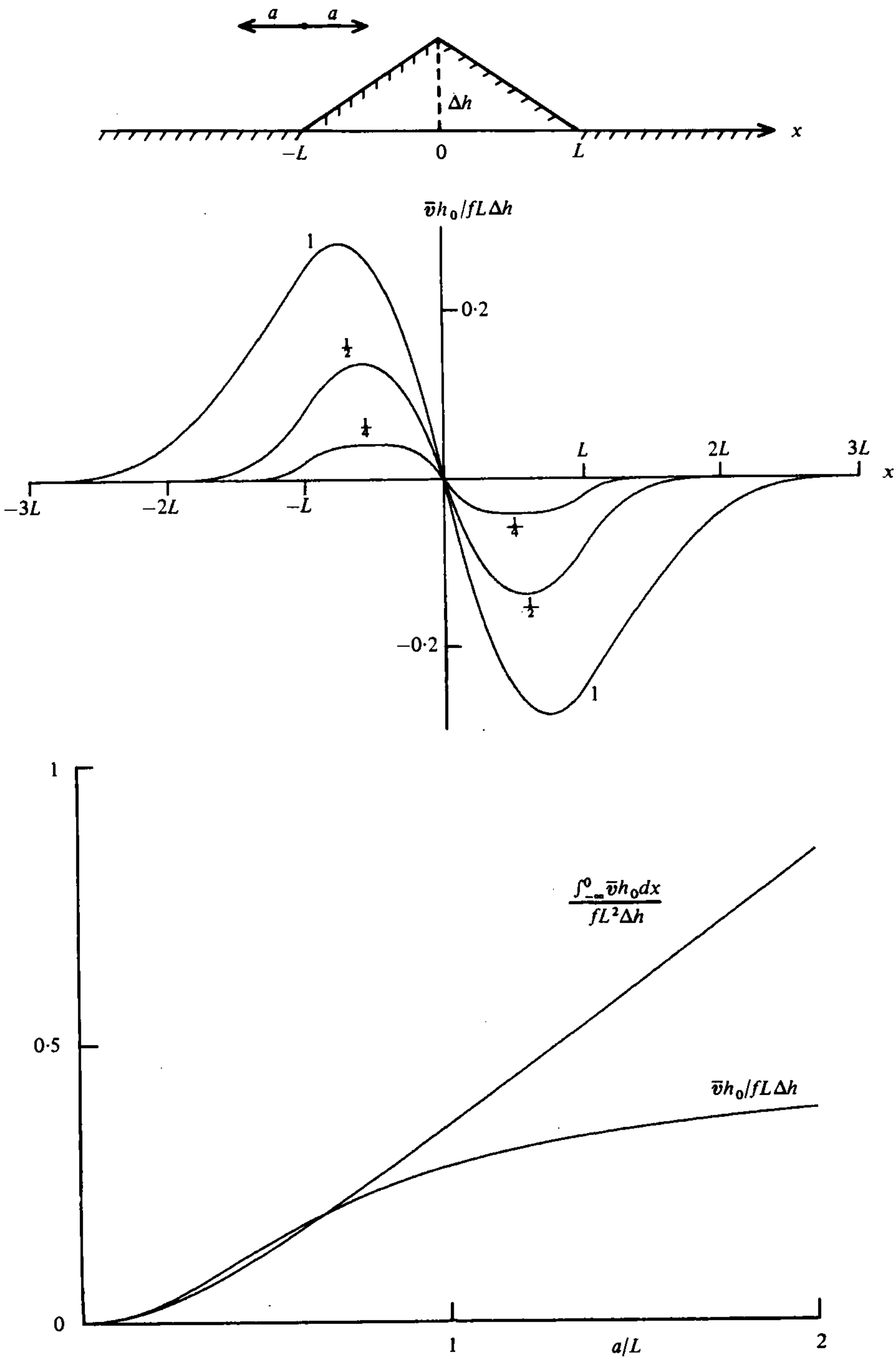

FIGURE 7. (*a*) Slope cross-section. (*b*) Mean longshore currents $\bar{v}$ for $a/L = \frac{1}{4}, \frac{1}{2}, 1$. (*c*) Dependence of maximum mean current $\bar{v}$ and transport $M = \int_{-\infty}^{0} \bar{v}\, dx$ on a/L.

where $\omega \equiv \partial v_1/\partial x - \partial u_1/\partial y$, $D/Dt \equiv \partial/\partial t + \mathbf{v}_0 . \nabla$. Vorticity is generated by the greater effect of the main bottom stress $\boldsymbol{\tau}_0$ where the water is shallower, and decays owing to the associated bottom stress torque curl $\boldsymbol{\tau}_1$. If the bottom stress is linear, i.e. $\boldsymbol{\tau}_1 = \mathbf{v}_1$, and F is small, then the solution of (7.7) for a particular fluid column P is

$$\omega_p = f(h_1 - \overline{h_1}^P)/h_0 + \mathbf{k} . \overline{\nabla h_1 \wedge \mathbf{v}_0}^P / h_0$$

where the overbars denote averages in time for P. We know the fluid column trajectory, defined by $\mathbf{v}_0$, viz. an ellipse with principal axes a, b (say), with which we can suppose the x, y axes are aligned. Hence we can find the time average of ω at any position $\mathbf{x}_0$ (instantaneously $\omega = \omega_p$ when P is at $\mathbf{x}_0$):

$$\overline{\omega} = f(h_1 - \langle h_1 \rangle)/h_0 + \mathbf{k} . \langle \nabla h_1 \wedge \tilde{\mathbf{v}}_0 \rangle / h_0. \tag{7.8}$$

Here the spatial averages $\langle\ \rangle$ are taken over the ellipse $\{\mathbf{x}_0 + 2\mu(a\cos\phi, b\sin\phi)$: $0 \leqslant \mu \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi\}$; there is a weighting factor $(4\pi^2 ab\mu(1-\mu^2)^{\frac{1}{2}})^{-1}$ in the averaging, and $\tilde{\mathbf{v}}_0 = \mp \sigma\mu(-a\sin\phi, b\cos\phi)$ according to whether the ellipse is described clockwise ($-$) or anticlockwise ($+$). Over local shallower areas, the first contribution to $\overline{\omega}$ gives anticyclonic vorticity and the second gives vorticity in the sense of the current ellipse polarization. The first term dominates for large particle excursions ($\epsilon \gg 1$).

Equations (7.8) and (7.6), i.e. $\nabla . \overline{\mathbf{v}}_1 = 0$, determine the mean flow field, which on account of (7.8) has (dimensional) scales $\epsilon U \Delta h/h_0$, $Lf\Delta h/h_0$ respectively for small and large ϵ, in agreement with §§5(*c*) and 6, and with Zimmerman (1978). A qualitative picture for intermediate ϵ is furnished by cross-slope oscillations over a straight ridge (figure 7*a*; $b = 0$ giving a 'neutral' zero second term in (7.8)) and approximating $\langle\ \rangle$ by a spatial mean. Figures 7(*b*, *c*) sketch the spreading and intensification (towards a maximum $\frac{1}{2}fL\Delta h/h_0$) of the long-slope mean current as $\epsilon = a/L$ increases. Over a monotonic slope between infinite regions of different uniform depth, the mean current differs in continuing to increase proportionally to ϵ for large ϵ, as does its breadth (Huthnance 1973*a*). However, the ridge is more representative of real topography where individual features are of finite horizontal extent.

Zimmerman (1979) has emphasized that, unless particle excursions (ϵ) are small, the Lagrangian-mean flow cannot be obtained by adding a Stokes drift to the present Eulerian mean, even though both are small, $O(\epsilon\epsilon_0 U)$ and $O(\delta U)$ respectively (see also McIntyre 1980). However, for the total mean current, it is possible simply to add the short-scale (L) mean current of this section to the mean currents arising from the large scale ($\epsilon_0^{-1}L$) variations of h_0 and the flow $\mathbf{v}_0$, these being calculated as in §3 since $\epsilon\epsilon_0 \ll 1$.

8. Discussion

We have considered the determination of mean (rectified) currents due to oscillatory motion above a frictional bottom boundary layer in shallow rotating fluid. For small amplitude oscillations, the main result (3.8) is that the depth-averaged mass transport, which follows geostrophic contours, has a strength such that closed fluid circuits around geostrophic contours experience no net time-averaged frictional torque. Hence the mean current is independent of the (weak) friction magnitude and quadratic in oscillatory quantities. Given this mean current, the mean currents in the bottom boundary layer can be calculated (Johns & Dyke 1972) to complete the analogy with

Longuet-Higgins' (1953) mass transport calculations in a non-rotating system. However, with rotation (as opposed to without) the mean current is determined weakly throughout the water column (rather than strongly just near the bottom boundary layer) and by all points around a geostrophic (f/h) contour in combination (rather than locally). The weak determination implies a liability to control by external constraints, notably termination of an f/h contour at a sidewall which imposes zero mass transport. The torque constraint (3.8) may also be modified by mean momentum fluxes in the boundary layer, and the mean current may include added contributions from horizontal density gradients and wind stress forcing as indicated by (3.11).

In shallow seas of limited extent, sea surface slope may also appear as an externally imposed variable (at open boundaries) which affects the mean circulation. This point of view is natural when modelling such seas numerically. However, the sea surface elevation is a flow variable and thereby subject to internally imposed constraints, particularly if frictional forces are genuinely weak. It is related to the mass transport by (3.7), a severe restriction if $\nabla_H(f/h) = O(1)$ so that ψ is a function only of f/h. If a geostrophic contour enters at one open boundary and leaves at another, the difference in the mean level $\bar{\zeta}_2$ between the two ends is required to match the difference in $\overline{\frac{1}{2}\mathbf{v}_1 . \nabla_H \zeta_1^t}$. Any discrepancy in matching this internal constraint is liable to magnification by $O(F^{-1})$ in its effects on mean currents. If possible, it appears preferable to specify the normal component of the latter at open boundaries, bearing in mind that, with error $O(F)$, the mass transport should be geostrophic.

Mean current magnitudes pass from $O(U^2/f_0 L \Delta h/h_0)$ to $O(f_0 L \Delta h/h_0)$ (if L is a topographic length scale) as $\epsilon = U/f_0 L$ increases through unity. These estimates agree with those of Zimmerman (1978) for small random topography. The quadratic increase with particle excursion U/f_0 levels off when the mean relative vorticity approaches a value representing uniform potential vorticity.

The present analysis is restricted to 'organized' barotropic motion subject to weak friction or growth. It therefore does not cover (for example) turbulent eddy contributions to oceanic circulation (Rhines & Holland 1979), mean motions associated with internal waves (e.g. Bretherton 1969) or friction-dominated estuarine circulation (Ianniello 1977, 1979).

The weak friction assumption, in the context of tidal currents, means decay times of a day or more; a one-dimensional example (Huthnance 1973*a*, (2.9)) shows a departure from the weak friction limit of just the fraction F. The assumption is not unduly restrictive in many shallow sea contexts, and tends to be compatible with the small amplitude (ϵ) assumption made in all but §6. Representative conditions for $\epsilon \leqslant 1$, $F \leqslant 1$ are $U \leqslant 1$ m s^{-1}, $L \geqslant 10$ km and $h \geqslant 20$ m.

The small F, ϵ results depend on neither parameter to a first approximation, which tends to increase their usefulness. However, the simple examples of §5 show that the particular type of friction – as opposed to its magnitude F – does affect the resulting mean currents. (Exceptions to this are eddy viscosity models $\nu \propto (z+h)^r$, $1 < r < 2$, which approximate the linear bottom drag law). Hence numerical models attempting to simulate mean 'residual' currents should represent bottom stress and boundary layer forms accurately.

Even more important for numerical modelling is that the weak $O(F)$ torque balance which determines the residual current strength should not be upset by spurious contributions from inaccurately-represented pressure terms. Where possible (e.g.

non-divergent flows) it may be easiest to use an explicit vorticity equation (Roache 1976). In any case, the numerical scheme (e.g. Sadourny 1975) should conserve both (i) mass, since the pressure/torque separation of the momentum balance (3.6) depends on the non-divergence of $\mathbf{U}_L$, and (ii) vorticity;

$$\overline{\mathbf{v}_1 . \nabla_H \mathbf{v}_1} - f\mathbf{k} \wedge \overline{\mathbf{v}_1^t . \nabla_H \mathbf{v}_1} \rightarrow -\tfrac{1}{2}\nabla_H(\overline{\mathbf{v}_1^t . \nabla \zeta_1}) \quad (F \rightarrow 0)$$

is also crucial to the torque balance and is therefore required of the numerical representation of these terms. These conditions tend to disfavour calculating mean flows using 'tidal stress' terms 'imported' from a separate tide model.

If the above conditions are satisfied, the mathematical analysis of § 3 has a numerical analogue. That is, the mean current generated by the numerical model is governed by a numerical equivalent of (3.8). An essentially explicit quadratic dependence on the oscillatory motion follows, and one can expect relative errors of the oscillatory motion to cause (only) doubled relative errors in the mean currents. Greater errors are likely to arise from the numerical handling of the frictional stress and bottom boundary-layer form.

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